AN ADAPTIVE SCHEME TO SYNCHRONIZE DIFFERENT CHAOTIC SYSTEMS

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Abstract— This paper proposes an adaptive methodology to synchronize any chaotic system with unified chaotic systems, even if bounded disturbances are present. The proposed controller is composed of both variable proportional and adaptive control actions for guaranteeing the convergence of the residual synchronization error to zero in the presence of disturbances. Two possible modifications are considered: 1) only adaptive control action is implemented to overcome the well-known assumption of prior knowledge of upper bounds to compensate for the disturbances, and 2) the control gain of the proportional part is saturated, when the residual synchronization error has, practically, been removed. Lyapunov theory, in combination with Barbalat’s Lemma, is used to design the proposed controller. Experimental simulations are provided to show the effectiveness of the proposed controller and its advantages, when compared with a recent work in the literature.

Keywords—Adaptive synchronization, adaptive control, chaotic systems, Lyapunov methods.

1 Introduction

Encouraged by the discovery of the chaotic dynamics by Lorenz in 1963 [1], chaos synchronization has been studied over the last thirty years by several researchers [2-12]. This is due to its potential applications in numerous engineering problems ranging from living system applications [5] to non-living system applications [6]. For instance, chaos synchronization has been applied in electrical [7,8], biological [9], chemical [10], secure communication [11], and finance systems [12].

After Lorenz’ model, several other chaotic systems such as Chua [13,14], Rössler [15,16], Lü [17,18], Chen [19,20], Liu [21,22], finance [23], unified [24], etc. have been proposed, and a great number of techniques, such as linear, nonlinear, passivity based, adaptive, backstepping, and sliding control, among others, have been introduced to achieve its synchronization. See, for instance,[25-29] and the references therein.

Despite the large number of existing techniques, few papers have been devoted to the chaotic synchronization in the presence of uncertain parameters and bounded disturbances. In most of these studies [29-36], the main particularity is that the control law is not smooth, since the control law depends on a sign function which is discontinuous [30-34,36]. Also, the parameter adaptation law is not robust, because it lacks a leakage term [30,31-34,36]. Moreover, the control law uses the time derivative of the synchronization error [29], the master system parameters [29] and disturbances [35], and it ensures that the synchronization error is only uniformly bounded in the presence of uncertain parameters and disturbances [36].

Unfortunately, in spite of the relevance of the above mentioned works ([29-36]), they have some limitations. It is well known that the discontinuous feedback control raises theoretical and practical issues [37]. From the theoretical point of view, standard existence and uniqueness of solutions of differential equations are, in general, not applicable [38]. Furthermore, the validity of the Lyapunov analysis will have to be examined in a framework that does not require the right-hand side of the state equation to be locally Lipschitz, in the state variables, and piecewise continuous with respect to time [37-38]. Practical issues are associated with the imperfections of switching devices and delays leading to chattering, which result, for example, in low control accuracy and high heat losses in electrical power circuits [37]. On the other hand, it has been known since the early 1980s that nonrobust adaptive laws may suffer from parameter drift phenomenon [39], that is, the parameters drift to infinity with time. It is often due to the “pure” integral action of the adaptive law. Several leakage modifications to counteract this have been proposed since then [39,40]. Finally, it is a basic rule in chaos synchronization based cryptography that the details of the encryption algorithm are always known by the attacker [41]. Hence, the use of master system parameters in the control law is controversial.

On the other hand; recently, a large number of studies have focused on synchronization of unified chaotic systems for applications in secure communications [42]. The unified chaotic system is a three dimensional system that has a broad spectrum of chaotic behavior, which is associated with a scalar parameter used in its model. As the unified system can
display hyperchaos [43] and the master system parameter can be used as a modulator element in chaotic parameter modulation schemes [11], it seems adequate for chaotic communication applications, and, hence, this chaotic system will be employed to show the proposed design methodology.

Motivated by the aforementioned facts, in this paper we propose a new scheme to synchronize any chaotic systems with unified chaotic systems. Depending on the choice of some design parameters, the proposed controller can be used: 1) to synchronize any chaotic system with unified chaotic systems without a well-known assumption of prior knowledge of an upper bound on the disturbance; 2) to ensure that the synchronization error converges to a desire arbitrary neighborhood of the origin. The proposed controller has advantageous features, when compared with previous works [29-36], since it is smooth, it does not require the master parameter for implementation purposes, and it uses a e-modification robust adaptive law [39] for adjusting the unknown parameter. Hence, it does not present chattering or parameter drift. The design methodology is based on Lyapunov theory and Barbala’s Lemma [39] and ensures the convergence of the synchronization error to zero, even in the presence of uncertain system parameter and bounded disturbances.

2 Problem Formulation

Consider the problem of control chaotic systems described by the following differential equation

\[
\dot{x}_s = A(\beta) x_s + f_s(x_s) + d_s(x_s, t) + u
\]

where \( x_s \in \mathbb{R}^3 \) is the state of the slave system, \( u \in \mathbb{R}^3 \) is the control input, \( f_s(.) \) is a known map, \( d_s(.) \) is an unknown disturbance, \( \beta \) is a known parameter,

\[
A(\beta) = \begin{bmatrix}
-25\beta - 10 & 25\beta + 10 & 0 \\
28 - 35\beta & 29\beta - 1 & 0 \\
0 & 0 & -\frac{8 + \beta}{3}
\end{bmatrix}
\]

and

\[
f_s(x_s) = \begin{bmatrix}
0 \\
-x_{s1}x_{s3} \\
x_{s1}x_{s2}
\end{bmatrix}
\]

We assume that the following can be established.

**Assumption 1:** The right-hand side of (1) is piecewise continuous with respect to time and locally Lipschitzian with respect to \( x_s \), such that (1) has a unique solution globally in time for any given initial condition.

\[
\|d_s(x_s, t)\| \leq d_{s0}
\]

**Assumption 2:** On the region \( \mathbb{R}^3 \times [0, \infty) \)

\[
\|d_m(x_m, t)\| \leq d_{m0}
\]

where \( d_{s0} \) is a positive constant, such that \( d_{s0} < \overline{d}_s \) and \( \overline{d}_s \) is a known constant.

**Remark 1:** Assumption 2 is usual in synchronization of chaotic systems.

**Remark 2:** In the case that \( \beta = 0, \beta = 0.8, \) and \( \beta = 1 \), system (1) becomes the Lorenz, Lü, and Chen systems, respectively, when perturbations are not present. However, any chaotic system of the form \( \dot{x}_s = F(x_s) + \overline{d}_s(x_s, t) - A(\beta)f_s(x_s) \).

**Remark 3:** It should be noted that the conditions of existence and uniqueness of solutions of (1), introduced in Assumption 1, are a prerequisite to the Lyapunov-type arguments to be used in the stability analysis. Basically, it is necessary to show that the state trajectories do not escape to infinity in finite time [38].

We consider the master system as

\[
\dot{x}_m = A(\alpha)x_m + f_m(x_m) + d_m(t, x_m)
\]

where \( x_m \in \mathbb{R}^3 \), \( \alpha \) is a known parameter, \( d_m(.) \) is an unknown disturbance,

\[
A(\alpha) = \begin{bmatrix}
-25\alpha - 10 & 25\alpha + 10 & 0 \\
28 - 35\alpha & 29\alpha - 1 & 0 \\
0 & 0 & -\frac{8 + \alpha}{3}
\end{bmatrix}
\]

and

\[
f_m(x_m) = \begin{bmatrix}
0 \\
-x_{m1}x_{m3} \\
x_{m1}x_{m2}
\end{bmatrix}
\]

**Assumption 3:** The parameter \( \alpha \) is upper bounded by a known positive constant \( \overline{\alpha} \), such that \( \overline{\alpha} > \alpha \).

**Assumption 4:** On the region \( \mathbb{R}^3 \times [0, \infty) \)

\[
\|d_m(x_m, t)\| \leq d_{m0}
\]

where \( d_{m0} \) is a positive constant, such that \( d_{m0} < \overline{d}_m \) and \( \overline{d}_m \) is a known constant.

**Assumption 5:** It is assumed that the right-hand side of (5) is piecewise continuous with respect to time and locally Lipschitzian with respect to \( x_m \), such that (5) has a unique solution globally in time.

Hence, our aim is to design a smooth feedback control \( u \), which depends on \( x_m, x_s, f_m, f_s \), and \( t \), but
does not depend on $\alpha$, such that the state $x_s$ of the slave chaotic system (1) tracks of the state $x_m$ of the master system (5), even in the presence of the disturbances $d_m$ and $d_r$.

Define the synchronization error $e(t) = x_s - x_m$.

Then, from (1) and (5), we obtain the synchronization error equation

$$
\dot{e} = A(\beta)e + f_s - f_m + d + u + \beta B x_m + \alpha C x_m
$$

where $d = d_s - d_m$, $C = -B$ and

$$
B = \begin{bmatrix}
-25 & 25 & 0 \\
-35 & 29 & 0 \\
0 & 0 & -1/3
\end{bmatrix}
$$

(9)

3 Adaptive Synchronization

In this sections, by using Lyapunov-like analysis in conjunction with Barbabal’s Lemma, it is shown that the synchronization error converges asymptotically to zero, in addition to boundedness, even in the presence of unknown master system parameters and disturbances. The proposed scheme is motivated by [40,44].

**Theorem 1:** Consider the slave (1) and master (5) chaotic systems, which satisfy Assumptions 1-5, the control law

$$
u = -(\alpha C x_m + \beta B x_m + f_s - f_m + A(\beta)e + Le + u_r)
$$

with

$$u_r = \frac{\gamma_0 e}{\lambda_{\text{min}}(K)} + \gamma_1 \exp(-\gamma_2 t)
$$

(11)

$$
\dot{\alpha} = -\alpha_0 \left[ \alpha - e^T KC x_m \right]
$$

(13)

where

$$
\dot{\beta} = \hat{\alpha} - \alpha - \alpha_1 \alpha
$$

(12)

The time derivative of (15) results

$$
\dot{V} = e^T P e + e_0 \alpha e + \hat{\alpha} \alpha_0^2 
$$

(16)

On the other hand, by using (9) and (11), the closed-loop synchronization error can be written as

$$
\dot{e} = -Le + \alpha C x_m + d - u_r
$$

(17)

By evaluating (16) along the trajectories of (13) and (17), we obtain

$$
\dot{V} = -e^T (L^T P + PL)e - e^T (P + P^T) C x_m \hat{\alpha}
$$

$$
- e^T (P + P^T) u_r + e^T (P + P^T) d
$$

$$
- \alpha_1 \left[ \hat{\alpha} \hat{\alpha} + e^T K C x_m \hat{\alpha} \right]
$$

(18)

Using now (14), (19) results

$$
\dot{V} = -e^T Q e + e^T K d - e^T K u_r - \alpha_1 \left[ \hat{\alpha} \hat{\alpha} + e^T [K]_e \right]
$$

(19)

which, by using (4), (8), and (14), (19) implies

$$
\dot{V} \leq -\gamma_3 \|e\|^2 + \gamma_4 \|e\| - \alpha_1 \left[ \hat{\alpha} \hat{\alpha} + e^T [K]_e \right]
$$

(20)

Since $(\alpha - \hat{\alpha})^2 = \alpha^2$, by binomial expansion, it can be established that

$$
\hat{\alpha} \hat{\alpha} = \left( \frac{\alpha}{2} + \frac{\alpha}{2} \right)^2 - \frac{\alpha^2}{2} = \frac{\alpha^2}{2}
$$

(21)

Thus, by employing (12)-(14) and (21), (20) implies

$$
\dot{V} \leq -\gamma_3 \|e\|^2 + \gamma_4 \|e\| - \alpha_1 \left[ \hat{\alpha} \hat{\alpha} + e^T [K]_e \right]
$$

(22)

At first, by using the fact $\gamma_0 > 0$, (22) can be re-written as

$$
\dot{V} \leq -\gamma_3 \|e\|^2 + \gamma_4 \|e\| - \alpha_1 \left[ \hat{\alpha} \hat{\alpha} + e^T [K]_e \right]
$$

(23)

Hence, $\dot{V} < 0$ outside the compact set

$$
\Omega_1 = \left\{ x : \left[ \begin{array}{c} e(t) \\ \hat{\alpha}(t) \end{array} \right] \in \mathbb{R}^{n+1} \right\}
$$

where $\alpha_p = (2\gamma_4/\alpha_1)^{1/2}$. Thus, since $\gamma_0$ and $\alpha_0$ are positive constants, by employing usual Lyapunov arguments [39], we concluded that $e(t)$ and $\hat{\alpha}(t)$ are uniformly bound. In addition, since $\gamma_0$ and $\alpha_0$ can be arbitrarily selected according to (14), $e(t)$ is uniformly ultimately bounded with ultimate bound $\alpha_p$.
In case $\gamma_o \geq 2\gamma_4$, (22) implies
$$V \leq -\gamma_3 ||x^0||^2 - \|y - y\| \leq \frac{2\gamma_4 \|\|y - y\| + \gamma_1 \exp(-\gamma_2 t)}{\gamma_3}$$
(24)

Define
$$\Omega_2 = \{x_a = \left[\begin{array}{c} e(t) \\ \tilde{e}(t) \end{array}\right] \in \mathbb{R}^{n+2} | \|e(t)\| \leq \gamma_1 \exp(-\gamma_2 t) \}$$
(25)

Note that the numerator in the key of (24) is greater than zero for $\|e\| > \gamma_1 \exp(-\gamma_2 t)$ (or $e \in \Omega_2$), hence
$$V \leq -\gamma_3 ||x^0||^2$$
(26)

Further, since $V$ is bounded from below and non-increasing with time, we have
$$\lim_{t \to \infty} \int_0^t \|\|e\| d\tau \leq \frac{V(0) - V_{\infty}}{\gamma_3} < \infty$$
(27)

where $\lim_{t \to \infty} V(t) = V_{\infty} < \infty$. Notice that, based on (17), with the bounds on $e, \tilde{e}, \gamma_2, \gamma_4$, and $\gamma_2$, it is also bounded. Thus, $V$ is uniformly continuous. Hence, by applying the Barbalat’s Lemma [39, p. 76], we conclude that $\lim_{t \to \infty} e(t) = 0$ for all $e \in \Omega_2$.

Once the synchronization error $e(t)$ has entered $\Omega_2$, it will remain in $\Omega_2$ forever, due to (25) and (26). Consequently, we conclude that $\lim_{t \to \infty} e(t) = 0$ holds in the large, i.e., whatever the initial value of $x_a^0$ (inside or outside $\Omega_2$).

**Remark 4:** Notice that in our formulation the control law does not depend on $\alpha$. This peculiarity can be explored by analog chaos-based secure communications systems, since the master system parameter $\alpha$ is not transparent to the receiver.

**Remark 5:** It is interesting to notice that the six first terms in the parenthesis of the right-hand side of (11) ensure the convergence of the synchronization error to an arbitrary neighborhood of the origin whose radius can be controlled, for instance, by the matrix $L$.

The last term, namely $u_r$, is designed to make the residual synchronization error asymptotically null with convergence rate controlled by $\gamma_2$.

**Corollary 1:** Consider the slave system (1), the master system (5), which satisfy Assumptions 1-5, and the control law (11) with $\gamma_2 = 0$. Then, the synchronization error $e(t)$ is uniformly ultimately bounded, with ultimate bound $\alpha_o = \gamma_4/\gamma_3$.

**Remark 6:** It should be pointed out that proposed controller can be used without any previous knowledge of upper bounds on disturbances, as established in Corollary 1, when only adaptive control action is used. In this case, the synchronization error converges to the neighborhood of the origin whose size can be reduced by setting high gains in the feedback loop.

**Corollary 2:** Consider the slave system (1), the master system (5), which satisfy Assumptions 1-5, and the control law (11) with $\gamma_2 = 0$. Then, the synchronization error $e(t)$ converges to the residual set
$$\Xi = \{e \in \mathbb{R}^2 | \|e(t)\| \leq \gamma_2 \}$$
where $\alpha_o = \exp(-\gamma_2 t)$ and $t_e$ is the time in which the exponential function in (12) is turned off.

**Remark 7:** Corollary 2 establish an interesting peculiarity of the proposed controller. The exponential function used in the control law can be turned off when the residual synchronization error has entered to any desired neighborhood of the origin. It is important to overcome numerical errors that can appear when the exponential function on the right-hand side of (12) has practically decayed to zero. Hence, the proposed controller does not present any chattering.

### 4 Simulations

The synchronization between chaotic systems which are not topologically equivalent, namely Bhalekar-Gejji and Chen systems, in the presence of disturbances is considered. Two possible scenarios are implemented: zero initial conditions and nonzero initial conditions for the synchronization error. A recent study [45] is used here for comparison.

Perturbed Bhalekar-Gejji system [46] can be written as
$$\dot{X}_s = \Lambda(\alpha, \mu, \eta)X_s + f_s(X_s) + d_s(X_s, t) + \tilde{\eta}$$
(30)

where $X_s \in \mathbb{R}^3$ is the state of the slave system, $\tilde{\eta} \in \mathbb{R}^3$ is the control input, $\Lambda(\alpha, \mu, \eta) = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & -\mu & 0 \\ 0 & 0 & -\eta \end{bmatrix}$,
$$f_s(X_s) = \begin{bmatrix} -X_s^2 \\ \mu X_s \\ X_s + e + e \end{bmatrix}, \quad \omega = -2.5667, \quad \mu = 10, \quad \eta = 1, \quad \epsilon = 27.3, \quad d_s(X_s, t)$$
is an unknown disturbance. Hence, the synchronization error equation can be written as
\[
\dot{\varepsilon} = \Lambda(\omega, \mu, \eta)\varepsilon + \left[\Lambda(\omega, \mu, \eta) - \Lambda(\alpha)\right] x_m + \tilde{f}_j - f_m + d + \Pi
\]
where \(\varepsilon = x_r - x_m\). To illustrate the advantages of the proposed methodology, the controller introduced in [45] is used here for comparison. Based on (31), the control law is chosen, similarly to [45], as
\[
\tilde{u} = -\left[\Lambda(\omega, \mu, \eta) - \Lambda(\alpha)\right] x_m - \tilde{f}_j + f_m
\]
On the other hand, to implement the proposed controller, it is observed that system (30) can be rewritten as
\[
\tilde{x}_s = \Lambda(\beta)\tilde{x}_r + f_s(\tilde{x}_s) + \Delta(\tilde{x}_s, t) + \tilde{u}
\]
where
\[
\Delta(\tilde{x}_s, t) = \left[\Lambda(\omega, \mu, \eta) - \Lambda(\beta)\right] \tilde{x}_s - f_s(\tilde{x}_s) + f_s(\tilde{x}_s, t)
\]
is a bounded disturbance. Hence, the proposed controller, equations (11)-(13), can be used, where \(e = \tilde{\varepsilon}, u = \tilde{u}\) and \(\tilde{x}_s\) is obtained from (30).

The design parameters are chosen as \(y_0 = 500, y_1 = 80, y_2 = 1.5, \alpha_1 = 20, \alpha_2 = 0.05, \alpha_3 = 10, \alpha_4 = 0, P = \text{diag}(0.001, 10, 5), \) and \(L = \text{diag}(10, 10, 5)\). To check the robustness of the proposed controller, the emergence of the disturbances (34) and (35) at \(t = 5\) s is considered.

\[
d_{m_1}(x_m, t) = \begin{bmatrix}
\sin(5t) + \cos(10t) \\
\sin(5t) \ln\left(0.5x_{m1}^2 + x_{m2}^2 + x_{m3}^2\right) \\
3\sin(3t) + \cos(3t)
\end{bmatrix}
\]

\[
d_{m_2}(x_m, t) = \begin{bmatrix}
3\sin(t) \left(x_{m1}^2 + x_{m2}^2 + 5x_{m3}^2\right)^{1/2} \\
3\sin(2t) \\
\cos(10t)
\end{bmatrix}
\]

Also, the exponential function in (12) is turned off at \(t = 4\), when the synchronization error has practically been removed. To make a detailed comparison with [45], two scenarios are considered. First, it is assumed that the chaotic systems have the same initial conditions, i.e., \(x_m(0) = \tilde{x}_r(0) = [1.5 \ 2 \ 5]^T\). Second, it is considered that the chaotic systems have different initial conditions, i.e., \(x_m(0) = [1.5 \ 2 \ 5]^T\) and \(\tilde{x}_r(0) = [4 \ 8 \ 3]^T\). The synchronization error trajectories are given in Fig. 1. The associated control input efforts \(u(t)\) and \(\tilde{u}(t)\) are depicted in Fig. 2. Based on Fig. 1, it can be seen that the proposed controller is, practically, not affected by the advent of disturbances at \(t = 5\). However, the price paid is a large initial control effort under adverse initial conditions, as depicted in Fig. 2, which is due to the initial value of the robustifying term in (11). From simulations, it can be concluded that the proposed controller is efficient at controlling the Bakelar-Gejji system, even under adverse initial conditions and in the presence of a severe class of disturbances, in contrast to [45].

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Figure 1. Synchronization error trajectories: a) \(e(0) = 0\) , b) \(\tilde{e}(0) = [2.5 \ 6 \ -2]^T\).
Hence, the proposed controller does not present any chattering. A comparison with a recent study in the literature was performed to show the advantages and peculiarities of the proposed method under disturbances.

5 Conclusions

In this paper, we proposed a robust synchronization scheme to synchronize any chaotic system with unified chaotic systems in the presence of disturbances. Based on Lyapunov-like analysis using Barbalat’s Lemma, an adaptive controller is proposed to force the state of the slave system to converge, asymptotically, to the state of the master system. Two particular cases were considered: 1) only adaptive control action was used to overcome prior knowledge of upper bounds for the disturbances and ensure the convergence of the residual synchronization error to any arbitrary neighborhood of the origin by adjusting the gains in the feedback loop, and 2) the gain of the variable proportional control term was saturated by turning off the exponential function in the robustifying term of the control law, when the residual synchronization error has, virtually, converged to zero.

References

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